Oxford Cambridge and RSA

Thursday 15 October 2020 – Afternoon

A Level Further Mathematics B (MEI)

Y421/01 Mechanics Major

Time allowed: 2 hours 15 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **120**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

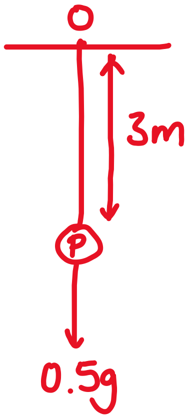
Section A (35 marks)

Answer **all** the questions.

1 A particle P of mass 0.5 kg is attached to a fixed point O by a light elastic string of natural length 3 m and modulus of elasticity 75 N.

P is released from rest at O and is allowed to fall freely.

Determine the length of the string when P is at its lowest point in the subsequent motion. **[5]**



$T = \frac{\lambda x}{L}$ $\quad EPE = \frac{\lambda x^2}{2L}$ $\quad GPE = mgh$

At P's lowest point:

$GPE = 0.5g(3 + x) = 14.7 + 4.9x$

x: Extension

$EPE = \frac{75x^2}{2(3)} = 12.5x^2$

Energy start = Energy End

$EPE = GPE$

$12.5x^2 = 14.7 + 4.9x$

$12.5x^2 - 4.9x - 14.7 = 0$

$$x = \frac{-(-4.9) \pm \sqrt{(-4.9)^2 - 4(12.5)(-14.7)}}{2(12.5)}$$

$x = 1.298\ldots$ $\qquad x \neq -0.906\ldots$ since $x > 0$.

$L + x = 3 + 1.298\ldots = 4.298\ldots \approx 4.30\text{ m}$ (3sf)

$\boxed{\therefore \text{Length of String} = 4.30\text{ m}}$

2 A student conducts an experiment by first stretching a length of wire and fixing its ends. The student then plucks the wire causing it to vibrate. The frequency of these vibrations, f, is modelled by the formula

$$f = kC^{\alpha}l^{\beta}\sigma^{\gamma},$$

where C is the tension in the wire,
l is the length of the stretched wire,
σ is the mass per unit length of the stretched wire and
k is a dimensionless constant.

Use dimensional analysis to find α, β and γ. **[5]**

$[f] = T^{-1}$

$[C] = MLT^{-1}$

$[\sigma] = ML^{-1}$

$T^{-1} = (MLT^{-2})^{\alpha} L^{\beta} (ML^{-1})^{\gamma} = M^{\alpha+\gamma} L^{\alpha+\beta-\gamma} T^{-2\alpha}$

① $\alpha + \gamma = 0$

② $\alpha + \beta - \gamma = 0$

③ $-2\alpha = -1 \Rightarrow \therefore \alpha = \frac{-1}{-2} = \frac{1}{2}$

$\alpha = \frac{1}{2} \Rightarrow$ ① $\gamma = -\alpha = -\frac{1}{2}$

$\alpha = \frac{1}{2}, \gamma = -\frac{1}{2} \Rightarrow$ ② $\beta = \gamma - \alpha = -\frac{1}{2} - \frac{1}{2} = -1$

$\boxed{\therefore \alpha = \frac{1}{2}, \beta = -1, \gamma = -\frac{1}{2}}$

3 The vertices of a triangular lamina, which is in the x–y plane, are at the origin O and the points $A(2,3)$ and $B(-2,1)$.

Forces $2\mathbf{i}+\mathbf{j}$ and $-3\mathbf{i}+2\mathbf{j}$ are applied to the lamina at A and B, respectively, and a force $\mathbf{F}$, whose line of action is in the x–y plane, is applied at O.

The three forces form a couple.

(a) Determine the magnitude and the direction of $\mathbf{F}$. **[4]**

(b) Determine the magnitude and direction of the additional couple that must be applied to the lamina in order to keep it in equilibrium. **[3]**

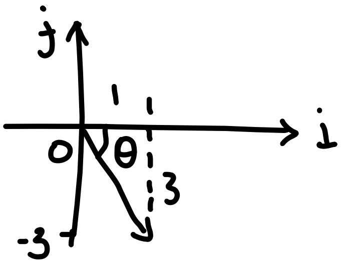
(a.) $\underline{F} = a\underline{i} + b\underline{j}$

$\underline{i}: a + 2 - 3 = 0 \Rightarrow \therefore a = 1$

$\underline{j}: b + 1 + 2 = 0 \Rightarrow \therefore b = -3$

$\therefore \underline{F} = \underline{i} - 3\underline{j}$

$|\underline{F}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$



$\tan\theta = \frac{3}{1} = 3$

$\therefore \theta = \tan^{-1}(3) = 71.56\ldots \approx 71.6^\circ$ (3sf)

$\boxed{\therefore \text{Magnitude} = \sqrt{10}, \text{Direction} = 71.6^\circ \text{ below the horizontal.}}$

(b.) $M(O): 3(2) + 2(2) - 2(1) - 1(3) = 5$

$\boxed{\therefore \text{Magnitude} = 5, \text{Direction} = \text{Anti-clockwise}}$

4 A particle P moves so that its position vector $\mathbf{r}$ at time t is given by

$\mathbf{r} = (5 + 20t)\mathbf{i} + (95 + 10t - 5t^2)\mathbf{j}$.

(a) Determine the initial velocity of P. **[3]**

At time $t = T$, P is moving in a direction perpendicular to its initial direction of motion.

(b) Determine the value of T. **[3]**

(c) Determine the distance of P from its initial position at time T. **[4]**

(a.) $\underline{v} = \dfrac{d\underline{r}}{dt} = 20\underline{i} + (10 - 10t)\underline{j}$

$t = 0 \Rightarrow \underline{v} = 20\underline{i} + (10 - 10(0))\underline{j} = 20\underline{i} + 10\underline{j}$

$\boxed{\therefore \text{Initial Velocity} = 20\underline{i} + 10\underline{j}}$

(b.) $t = T$:

$\begin{pmatrix} 20 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 10 - 10T \end{pmatrix} = 0$

$20(20) + 10(10 - 10T) = 0$

$400 + 100 - 100T = 0$

$500 = 100T$

$\therefore T = \dfrac{500}{100} = 5$ $\boxed{\therefore T = 5}$

(c.) $t = 0 \Rightarrow \underline{r} = (5 + 20(0))\underline{i} + (95 + 10(0) - 5(0)^2)\underline{j}$

$\therefore \underline{r} = 5\underline{i} + 95\underline{j}$

$t = T = 5 \Rightarrow \underline{r} = (5 + 20(5))\underline{i} + (95 + 10(5) - 5(5)^2)\underline{j}$

$\therefore \underline{r} = 105\underline{i} + 20\underline{j}$

Displacement Vector $= \begin{pmatrix} 105 \\ 20 \end{pmatrix} - \begin{pmatrix} 5 \\ 95 \end{pmatrix} = \begin{pmatrix} 100 \\ -75 \end{pmatrix} = 100\underline{i} - 75\underline{j}$

$|100\underline{i} - 75\underline{j}| = \sqrt{100^2 + (-75)^2} = 125$ $\boxed{\therefore \text{Distance} = 125\,m}$

5 A car of mass 900 kg moves along a straight level road. The power developed by the car is constant and equal to 60 kW. The resistance to the motion of the car is constant and equal to 1500 N.

At time t seconds the velocity of the car is denoted by $v\,\text{m s}^{-1}$. Initially the car is at rest.

(a) Show that $\frac{3v}{5}\frac{dv}{dt} = 40 - v$. **[3]**

(b) Verify that $t = 24\ln\left(\frac{40}{40-v}\right) - \frac{3}{5}v$. **[5]**

(a.) $P = fv \Rightarrow f = \frac{P}{v}$

Driving Force, Df $= \frac{60000}{v}$

$F = ma \equiv F = m\frac{dv}{dt}$

$\frac{60000}{v} - 1500 = 900\frac{dv}{dt}$

$\times \frac{v}{300}$ $\quad \times \frac{v}{300}$ $\quad \times \frac{v}{300}$

$200 - 5v = 3v\frac{dv}{dt}$

$\div 5$ $\quad \div 5$ $\quad \div 5$

$\boxed{\therefore \frac{3v}{5}\frac{dv}{dt} = 40 - v}$ as required

(b.) $v = 0 \Rightarrow t = 24\ln\left(\frac{40}{40-0}\right) - \frac{3}{5}(0) = 24\ln 1 = 0$

$\frac{d}{dv}\left[\frac{40}{40-v}\right] = \frac{d}{dv}\left[40(40-v)^{-1}\right] = 40(40-v)^{-2}$

using product rule $\quad u = 40 \quad v = (40-v)^{-1}$

$u' = 0 \quad v' = -(-1)(40-v)^{-2} = (40-v)^{-2}$

$\frac{dt}{dv} = 24\left[\frac{1}{\left(\frac{40}{40-v}\right)}\right] \times (40)(40-v)^{-2} - 0(40-v)^{-1} - \frac{3}{5}$

$$\frac{dt}{dv} = 24\left(\frac{\cancel{40-v}}{\cancel{40}}\right)\left(\frac{\cancel{40}}{(40-v)^{\cancel{2}}}\right) - \frac{3}{5}$$

$$\frac{dt}{dv} = \frac{24}{40-v} - \frac{3}{5} = \frac{24(5) - 3(40-v)}{5(40-v)} = \frac{3v}{5(40-v)}$$

$$\therefore \frac{3v}{5}\frac{dv}{dt} = 40-v$$ as shown in (a)

$$\boxed{\therefore t = 24\ln\left(\frac{40}{40-v}\right) - \frac{3}{5}v}$$

as required

Section B (85 marks)

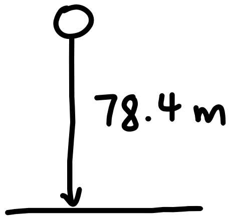
6 A small ball of mass m kg is held at a height of 78.4 m above horizontal ground. The ball is released from rest, falls vertically and rebounds from the ground. The coefficient of restitution between the ball and ground is e.

The ball continues to bounce until it comes to rest after 6 seconds.

(a) Determine the value of e. **[8]**

(b) Given that the magnitude of the impulse that the ground exerts on the ball at the first bounce is 23.52 Ns, determine the value of m. **[2]**

(a.)



$S = 78.4$
$U = 0$
$V = X$
$A = g$
$T = t_1$

$s = ut + \frac{1}{2}at^2$

$78.4 = 0 + \frac{1}{2}gt_1^2$

$t_1^2 = \frac{78.4(2)}{9.8} = 16$

$\therefore t_1 = \sqrt{16} = 4s$

$v = u + at$

$v_1 = 0 + gt_1 = 9.8(4) = 39.2$

$v = eu$

$v_2 = e(39.2) = 39.2e$

$S = 0$
$U = 39.2e$
$V = X$
$A = -g$
$T = t_2$

$s = ut + 0.5at^2$

$0 = 39.2et_2 - 0.5gt_2^2$

$4.9t_2^2 - 39.2et_2 = 0$

$t_2(4.9t_2 - 39.2e) = 0$

$t_2 \neq 0$ $\quad \therefore t_2 = \frac{39.2e}{4.9} = 8e$

Series: $4 + 8e + 8e^2 + \dots = 6$

$S_\infty = \frac{a}{1-r}$

if ball continues bouncing

$4 + 8e(1 + e + \dots) = 6$

$4 + 8e\left(\frac{1}{1-e}\right) = 6$

$\frac{8e}{1-e} = 6-4$

$8e = 2(1-e)$

$8e = 2 - 2e$

$10e = 2$

$\therefore e = \frac{2}{10} = \frac{1}{5} = 0.2$

$\boxed{\therefore e = 0.2}$

(b.) $|I| = 23.52\ Ns$

$I = m(v-u)$

$u = -39.2$

$v = 39.2e = 39.2(0.2) = 7.84$

$23.52 = m(7.84 - -39.2)$

$23.52 = 47.02$

$m = \frac{23.52}{47.04} = 0.5$

$\boxed{\therefore m = 0.5\,kg}$

7

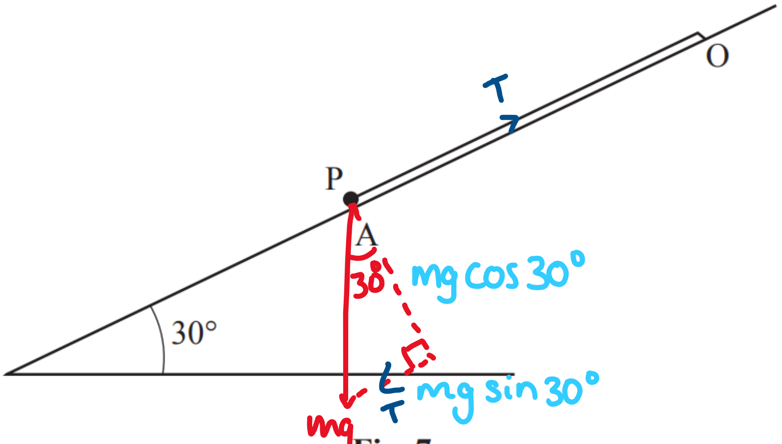


Fig. 7

A particle P of mass m is attached to one end of a light elastic string of natural length $6a$ and modulus of elasticity $3mg$. The other end of the string is fixed to a point O on a smooth plane, which is inclined at an angle of 30° to the horizontal. The string lies along a line of greatest slope of the plane and P rests in equilibrium on the inclined plane at a point A, as shown in Fig. 7.

P is now pulled a further distance $2a$ down the line of greatest slope through A and released from rest. At time t later, the displacement of P from A is x, where the positive direction of x is down the plane.

(a) Show that, until the string slackens, x satisfies the differential equation

$$\frac{d^2x}{dt^2} + \frac{gx}{2a} = 0.$$ **[6]**

(b) Determine, in terms of a and g, the time at which the string slackens. **[5]**

(c) Find, in terms of a and g, the speed of P when the string slackens. **[2]**

(a.) Hooke's law: $T = \frac{\lambda e}{l}$ $\Rightarrow \therefore T = \frac{3mge}{6a} = \frac{mge}{2a}$

$T = mg\sin 30° = \frac{mg}{2}$

$\frac{\cancel{mg}e}{\cancel{2}a} = \frac{\cancel{mg}}{\cancel{2}}$ $\Rightarrow \therefore e = a$

$F = ma = m\dfrac{d^2x}{dt^2}$

$mg\sin 30° - T = m\dfrac{d^2x}{dt^2}$

$\cancel{m}g\sin 30° - \dfrac{\cancel{m}g(a+x)}{2a} = \cancel{m}\dfrac{d^2x}{dt^2}$

$\dfrac{g}{2} - \dfrac{(ga+gx)}{2a} = \dfrac{d^2x}{dt^2}$

$\times a$

$\dfrac{g\cancel{a} - g\cancel{a} - gx}{2a} = \dfrac{d^2x}{dt^2}$

$\boxed{\therefore \dfrac{d^2x}{dt^2} + \dfrac{gx}{2a} = 0}$ as required

(b.) $x = A\cos\sqrt{\dfrac{g}{2a}}\,t + B\sin\sqrt{\dfrac{g}{2a}}\,t$

$t=0, x=2a \Rightarrow 2a = A + 0B \Rightarrow \therefore A = 2a$

$t=0, \dot{x}=0 \Rightarrow \therefore B=0$

$\therefore x = 2a\cos\sqrt{\dfrac{g}{2a}}\,t$

Slacks when $x=-a \Rightarrow -a = 2a\cos\left(\sqrt{\dfrac{g}{2a}}\,t\right)$

$\cos^{-1}\left(-\dfrac{\cancel{a}}{2\cancel{a}}\right) = \sqrt{\dfrac{g}{2a}}\,t$

$\dfrac{2\pi}{3} = t\sqrt{\dfrac{g}{2a}}$

$t = \dfrac{2\pi}{3} \div \left(\dfrac{g}{2a}\right)^{1/2} = \dfrac{2\pi}{3}\sqrt{\dfrac{2a}{g}}$

$\boxed{\therefore t = \dfrac{2\pi}{3}\sqrt{\dfrac{2a}{g}}}$

(c.) $v^2 = \omega^2 (A^2 - x^2)$

$$v^2 = \frac{g}{2a}\left((2a)^2 - (-a)^2\right) = \frac{g}{2a}\left(4a^2 - a^2\right) = \frac{3ga}{2}$$

$$\boxed{\therefore v = \sqrt{\frac{3ga}{2}}}$$

8 [In this question, you may use the fact that the volume of a right circular cone of base radius r and height h is $\frac{1}{3}\pi r^2 h$.]

(a) By using integration, show that the centre of mass of a uniform solid right circular cone of height h and base radius r is at a distance $\frac{3}{4}h$ from the vertex. **[5]**

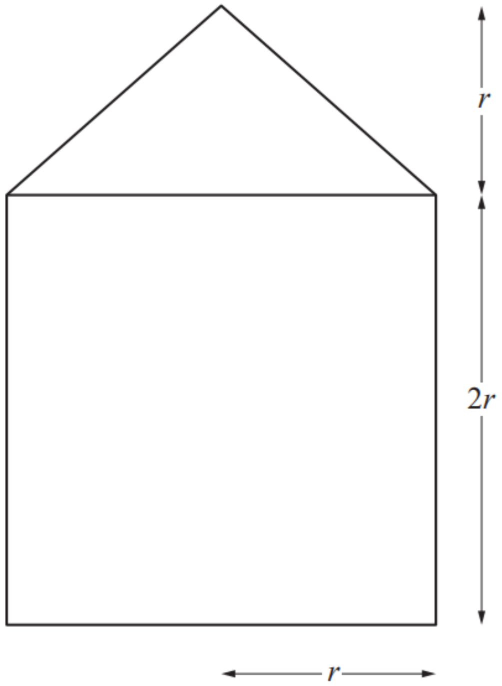


Fig. 8

Fig. 8 shows the side view of a toy formed by joining a uniform solid circular cylinder of radius r and height $2r$ to a uniform solid right circular cone, made of the same material as the cylinder, of radius r and height r.

The toy is placed on a horizontal floor with the curved surface of the cone in contact with the floor.

(b) Determine whether the toy will topple. **[7]**

(c) Explain why it is not necessary to know whether the floor is rough or smooth in answering part **(b)**. **[1]**

(a.) $V\bar{x} = \pi \int x y^2 \, dx$

$y = \frac{r}{h}x$

$V\bar{x} = \pi \int_0^h x\left(\frac{r}{h}x\right)^2 dx$

Since $V = \frac{1}{3}\pi r^2 h$: $\frac{\pi}{3} r^2 h \bar{x} = \pi \int_0^h \frac{x^3 r^2}{h^2} dx$

$$\frac{\pi}{3} r^2 h \bar{x} = \frac{\pi r^2}{h^2}\left[\frac{x^4}{4}\right]_0^h$$

$$\frac{\cancel{\pi}}{3} \cancel{r^2} h \bar{x} = \frac{\cancel{\pi} \cancel{r^2}}{h^2}\left(\frac{h^4}{4} - 0\right)$$

$\frac{h}{3}\bar{x} = \frac{h^2}{4}$ $\quad \boxed{\therefore \bar{x} = \frac{3h}{4}}$ as required

(b.) $\left(\frac{1}{3}\pi r^3 + \pi r^2(2r)\right) x_G = \frac{1}{3}\pi r^3\left(\frac{3r}{4}\right) + 2r\left(\pi r^2 (2r)\right)$

$$\frac{7}{3}\cancel{\pi r^3} x_G = \frac{1}{4}\cancel{\pi} r^{\cancel{4}} + 4\cancel{\pi} r^{\cancel{4}}$$

$$\frac{\cancel{7}}{\cancel{3}} x_G = \frac{17}{4} r$$

$\times \frac{\cancel{3}}{\cancel{7}}$ $\quad \times \frac{3}{7}$ $\quad \therefore x_G = \frac{51}{28} r$

$$\cos 45^\circ = \frac{x}{\left(\frac{51}{28} r\right)} \Rightarrow x = \frac{51}{56}\sqrt{2} r$$

Slant height of cone is $r\sqrt{2}$.

$$\boxed{\frac{51}{56} r\sqrt{2} < r\sqrt{2} \Rightarrow \therefore \text{Doesn't topple}}$$

(c.) Moment of frictional force about any point of contact with the horizontal floor is zero and so has no effect on the stability of the toy.

9

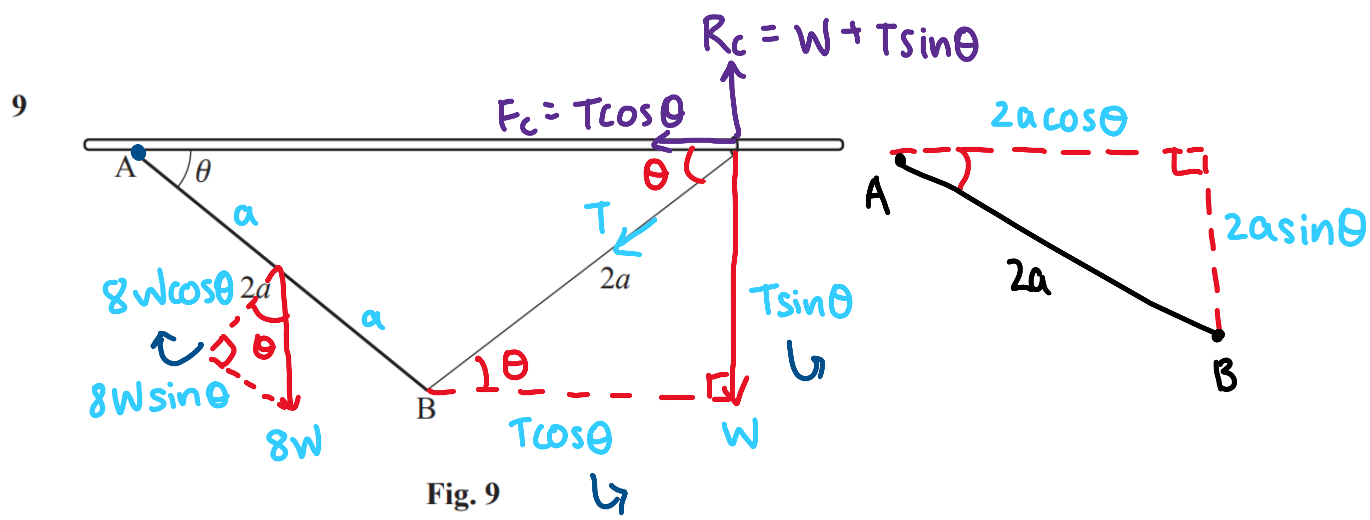


Fig. 9

Fig. 9 shows a uniform rod AB of length $2a$ and weight $8W$ which is smoothly hinged at the end A to a point on a fixed horizontal rough bar. A small ring of weight W is threaded on the bar and is connected to the rod at B by a light inextensible string of length $2a$. The system is in equilibrium with the rod inclined at an angle θ to the horizontal.

(a) Determine, in terms of W and θ, the tension in the string. **[4]**

It is given that, for equilibrium to be possible, the greatest distance the ring can be from A is $2.4a$.

(b) Determine the coefficient of friction between the bar and the ring. **[6]**

(a) $M(A)$: $8W(a\cos\theta) = (2a\cos\theta)(T\sin\theta) + (2a\sin\theta)(T\cos\theta)$

$8Wa\cos\theta = 2aT\sin\theta\cos\theta + 2aT\sin\theta\cos\theta$

$8W\cancel{a}\,\cancel{\cos\theta} = 4\cancel{a}T\sin\theta\,\cancel{\cos\theta}$

$T = \dfrac{8Wa}{4a\sin\theta} = 2W\operatorname{cosec}\theta$ $\boxed{\therefore T = 2W\operatorname{cosec}\theta}$

(b) $R_c = W + T\sin\theta$, $F_c = T\cos\theta$

$F_r = \mu R$

$F_c \leq \mu R_c \Rightarrow 2\cancel{W}\cot\theta \leq \mu(3\cancel{W})$

$\cot\theta \leq \dfrac{3\mu}{2} \Rightarrow \tan\theta \geq \dfrac{2}{3\mu}$

Least value of tan gives greatest distance, $\therefore \tan\theta = \dfrac{2}{3\mu}$

$2.4a = 4a\cos\theta \Rightarrow \cos\theta = 0.6$

$\dfrac{2}{3\mu} = \dfrac{4}{3} \Rightarrow$ $\boxed{\therefore \mu = \frac{1}{2}}$

10

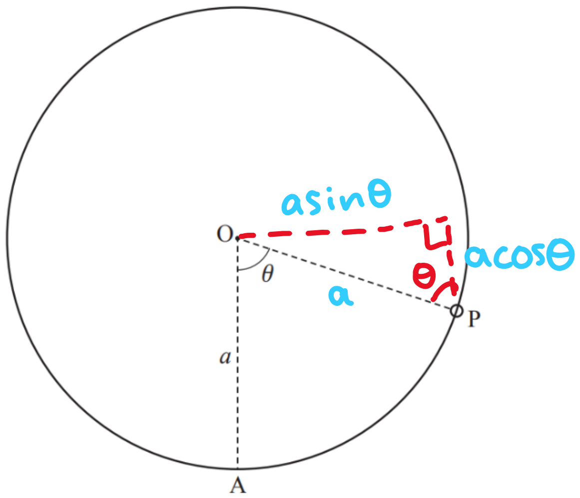


Fig. 10

Fig. 10 shows a small bead P of mass m which is threaded on a smooth thin wire. The wire is in the form of a circle of radius a and centre O. The wire is fixed in a vertical plane.

The bead is initially at the lowest point A of the wire and is projected along the wire with a velocity which is just sufficient to carry it to the highest point on the wire. The angle between OP and the downward vertical is denoted by θ.

(a) Determine the value of θ when the magnitude of the reaction of the wire on the bead is $\frac{7}{2}mg$. **[7]**

(b) Show that the angular velocity of P when OP makes an angle θ with the downward vertical is given by $k\sqrt{\frac{g}{a}}\cos\left(\frac{\theta}{2}\right)$, stating the value of the constant k. **[4]**

(c) Hence determine, in terms of g and a, the angular acceleration of P when θ takes the value found in part **(a)**. **[3]**

(a.) GPE $= mgh$, KE $= \frac{1}{2}mv^2$

At highest point: GPE $= mg(2a) = 2mga$, KE $= 0$

At angle θ: GPE $= mga(1-\cos\theta)$, KE $= \frac{1}{2}mv^2$

$2\cancel{m}ga = \cancel{m}ga(1-\cos\theta) + \frac{1}{2}\cancel{m}v^2$

$\times 2 \qquad \times 2 \qquad \times 2$

$4ga = 2ga(1-\cos\theta) + v^2$

$\therefore v^2 = 4ga - 2ga(1-\cos\theta) = 2ga(1+\cos\theta)$

$F_c = \frac{mv^2}{r}$

$R - mg\cos\theta = \frac{mv^2}{a}$

$R = mg\cos\theta + \frac{mv^2}{a} = mg\cos\theta + \frac{2mg\cancel{a}(1+\cos\theta)}{\cancel{a}}$

$\therefore R = mg(2 + 3\cos\theta)$

$\frac{7}{2}\cancel{mg} = \cancel{mg}(2 + 3\cos\theta)$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ $\boxed{\therefore \theta = \frac{\pi}{3}}$

(b.) $a^2\omega^2 = 2ga(1+\cos\theta)$

$2\cos^2\theta - 1 = \cos 2\theta \Rightarrow \therefore \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$a^2\omega^2 = 2ga\left(\cancel{1} + 2\cos^2\left(\frac{\theta}{2}\right) - \cancel{1}\right)$

$a^2\omega^2 = 4ga\cos^2\left(\frac{\theta}{2}\right)$

$\omega^2 = \frac{4g}{a}\cos^2\left(\frac{\theta}{2}\right)$

$\omega = \sqrt{\frac{4g}{a}\cos^2\left(\frac{\theta}{2}\right)}$ $\boxed{\therefore \omega = 2\sqrt{\frac{g}{a}}\cos\left(\frac{\theta}{2}\right)}$ k=2

(c.) $\frac{d\omega}{dt} = -\frac{1}{2}\left(2\sqrt{\frac{g}{a}}\right)\sin\left(\frac{\theta}{2}\right)\omega$

$\frac{d\omega}{dt} = -\sqrt{\frac{g}{a}}\sin\left(\frac{\pi}{6}\right)\left(2\sqrt{\frac{g}{a}}\cos\left(\frac{\pi}{6}\right)\right)$

$\boxed{\therefore \ddot{\theta} = -\frac{g\sqrt{3}}{2a}}$

11 Two uniform small smooth spheres A and B have equal radii and equal masses. The spheres are on a smooth horizontal surface. Sphere A is moving at an acute angle α to the line of centres, when it collides with B, which is stationary.

After the impact A is moving at an acute angle β to the line of centres. The coefficient of restitution between A and B is $\frac{1}{3}$.

(a) Show that $\tan\beta = 3\tan\alpha$. **[5]**

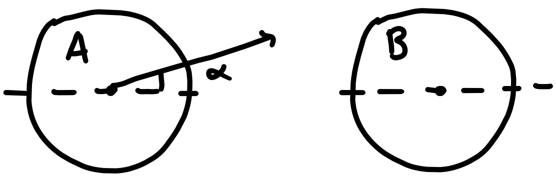
(b) Explain why the assumption that the contact between the spheres is smooth is needed in answering part **(a)**. **[1]**

It is given that A is deflected through an angle γ.

(c) Determine, in terms of α, an expression for $\tan\gamma$. **[2]**

(d) Determine the maximum value of γ. You do not need to justify that this value is a maximum. **[5]**

(a.)



$\rho = mv$

$v_B - v_A = e(u_A - u_B)$

$\cancel{m}u\cos\alpha = \cancel{m}v_A + \cancel{m}v_B$ ①

$v_B - v_A = \frac{1}{3}u\cos\alpha$ ②

① $v_A + \cancel{v_B} = u\cos\alpha$

② $-v_A + \cancel{v_B} = \frac{1}{3}u\cos\alpha$

$\cancel{2}v_A = \frac{2}{3}u\cos\alpha$

$\therefore v_A = \frac{1}{3}u\cos\alpha$

$\tan\beta = \frac{u\sin\alpha}{v_A} = \frac{\cancel{u}\sin\alpha}{\frac{1}{3}\cancel{u}\cos\alpha} = 3\tan\alpha$

$\boxed{\therefore \tan\beta = 3\tan\alpha}$

(b.) The component of the velocity of A perpendicular to the line of centres doesn't change.

(c.) $\tan\gamma = \tan(\beta - \alpha) = \dfrac{3\tan\alpha - \tan\alpha}{1 + 3\tan\alpha\tan\alpha}$

$$\boxed{\therefore \tan\gamma = \frac{2\tan\alpha}{1 + 3\tan^2\alpha}}$$

(d.) Differentiating $\tan\gamma = \dfrac{2\tan\alpha}{1+3\tan^2\alpha}$: using quotient rule

$$\sec^2\gamma \frac{d\gamma}{d\alpha} = \frac{(1+3\tan^2\alpha)(2\sec^2\alpha) - (2\tan\alpha)(6\tan\alpha\sec^2\alpha)}{[1+3\tan^2\alpha]^2} = 0$$

equate to 0

getting $\Rightarrow$ $2\cancel{\sec^2\alpha}(1 + 3\tan^2\alpha - 6\tan^2\alpha) = 0$

$1 + 3\tan^2\alpha - 6\tan^2\alpha$

$$1 + 3\tan^2\alpha - 6\tan^2\alpha = 0$$

$$3\tan^2\alpha = 1$$

$$\tan^2\alpha = \frac{1}{3}$$

$$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$$

$$\tan\gamma = \frac{2\left(\frac{1}{\sqrt{3}}\right)}{1 + 3\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\sqrt{3}}{3}$$

$$\therefore \gamma = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$\boxed{\therefore \text{Maximum value of } \gamma = \frac{\pi}{6}}$$

12

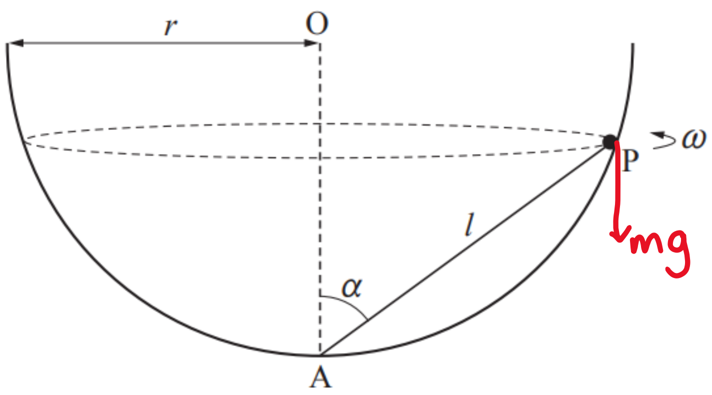


Fig. 12

Fig. 12 shows a hemispherical bowl. The rim of this bowl is a circle with centre O and radius r. The bowl is fixed with its rim horizontal and uppermost.

A particle P, of mass m, is connected by a light inextensible string of length l to the lowest point A on the bowl and describes a horizontal circle with constant angular speed ω on the smooth inner surface of the bowl.

The string is taut, and AP makes an angle α with the vertical.

(a) Show that the normal contact force between P and the bowl is of magnitude $mg + 2mr\omega^2\cos^2\alpha$. **[9]**

(b) Deduce that $g < r\omega^2(k_1 + k_2\cos^2\alpha)$, stating the value of the constants k_1 and k_2. **[3]**

(a.) $\cancel{r^2} = \cancel{r^2} + l^2 - 2rl\cos\alpha$

$l(l - 2r\cos\alpha) = 0$

$l \neq 0$ $\quad \therefore l = 2r\cos\alpha$

$\sin\alpha = \dfrac{x}{l} \Rightarrow x = l\sin\alpha = 2r\cos\alpha\sin\alpha$

Resolving vertically for P:

$R\sin\theta = T\cos\alpha + mg$

$R\cos\theta + T\sin\alpha = m(2r\cos\alpha\sin\alpha)\omega^2$

$\cos\theta = \dfrac{2r\cos\alpha\sin\alpha}{r} = 2\cos\alpha\sin\alpha$

$$R(1-2\cos^2\alpha) = T\cos\alpha + mg$$

$$R(2\cos\alpha\sin\alpha) + T\sin\alpha = 2mr\omega^2\cos\alpha\sin\alpha$$

$$T = 2mr\omega^2\cos\alpha - 2R\cos\alpha$$

$$R(1-2\cos^2\alpha) = (2mr\omega^2\cos\alpha - 2R\cos\alpha)\cos\alpha + mg$$

$$R - 2R\cos^2\alpha = 2mr\omega^2\cos^2\alpha - 2R\cos^2\alpha + mg$$

$$\boxed{\therefore R = mg + 2mr\omega^2\cos^2\alpha}$$ as required

(b.) $T = 2m\cos\alpha\,(r\omega^2 - g - 2r\omega^2\cos^2\alpha)$

$$T > 0 \Rightarrow r\omega^2 - g - 2r\omega^2\cos^2\alpha > 0$$

$$\boxed{\begin{array}{l}\therefore g < r\omega^2(1-2\cos^2\alpha) \\ \therefore k_1 = 1,\ k_2 = -2\end{array}}$$